# Universal entanglement timescale for Rényi entropies

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Recently it was shown that the growth of entanglement in an initially separable state, as measured by the purity of subsystems, can be characterized by a timescale that takes a universal form for any Hamiltonian. We show that the same timescale governs the growth of entanglement for all Rényi entropies. Since the family of Rényi entropies completely characterizes the entanglement of a pure bipartite state, our timescale is a universal feature of bipartite entanglement. The timescale depends only on the interaction Hamiltonian and the initial state.

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## I. INTRODUCTION

Composite quantum systems exhibit correlations among subsystems which cannot be explained in terms of classical probabilities. For pure states, these quantum correlations are known as entanglement. In this paper, we study how entanglement is generated by the mutual interactions among subsystems as the overall state evolves in time.

The time evolution of entanglement has become a focus in a variety of research fields. Its early study in quantum optical systems [1,2] has bloomed into a major area of research in many-body and condensed-matter systems [3–7], and conformal field theories dual to theories of quantum gravity [8–10]. For some classes of systems, general features have been found, including scaling laws [11,12] and generic linear growth [13–16].

The growth of entanglement is especially important in experimental systems where entanglement between the system and its environment leads to decoherence [17]. A complete understanding of the evolution of entanglement requires solving the dynamics of the overall state. This is often not feasible, including for decoherence where the Hamiltonian describing interactions with the environment is not known explicitly.

It is therefore interesting to ask what aspects of entanglement growth, if any, are shared by all quantum systems. Broad statements can be made in this direction with minimal assumptions about system dynamics by relying on special initial conditions instead.

To begin, bipartite entanglement between subsystems must be defined with respect to a partition of the system's degrees of freedom, represented as a fixed factorization of the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The Hamiltonian for the full system can be expressed as

$$H = \sum_n A_n \otimes B_n, \tag{1}$$

where each $A_n$ is an operator acting on subsystem $\mathcal{H}_A$, and each $B_n$ acts on $\mathcal{H}_B$. Any number of terms may be included as long as $H$ is Hermitian. Since the algebra of operators acting on $\mathcal{H}$ is isomorphic to the tensor product of subsystem algebras, any Hamiltonian can be represented this way [18].

Recently it was shown by Yang [19] that starting from a pure, unentangled state,

$$|\Psi(0)\rangle = |\psi(0)\rangle_A \otimes |\psi(0)\rangle_B , \tag{2}$$

the growth of entanglement under the unitary evolution generated by (1) is characterized by a universal timescale,

$$T_{\text{ent}} = \left[ \sum_{n,m} (\langle A_n A_m \rangle - \langle A_n \rangle \langle A_m \rangle)(\langle B_n B_m \rangle - \langle B_n \rangle \langle B_m \rangle) \right]^{-\frac{1}{2}}. \tag{3}$$

Here the expectation values are taken in the initial state. The timescale is universal in the sense that it takes this form for any quantum system that satisfies the requirements (1) and (2). The entanglement timescale was derived by studying one particular measure of the entanglement between subsystems $A$ and $B$, namely, the purity $P(\rho_A) = \text{tr}_A \rho_A^2$ of the reduced density matrix $\rho_A = \text{tr}_B \rho$. By the assumption (2), the purity is initially maximal so that its dynamics are governed at lowest order in $t$ by $d^2P/dt^2$. The second derivative is proportional to $T_{\text{ent}}^{-2}$ which is entirely determined by the expectation values of the interaction Hamiltonian operators in the initial state.

In this paper, we show that the same entanglement timescale (3) governs the growth of entanglement as measured by the entire family of quantum Rényi entropies [20],

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln \text{tr}_A \rho_A^\alpha, \tag{4}$$

where $\alpha$ is taken to be a positive integer. As a family, the Rényi entropies provide complete information about the eigenvalue distribution of the reduced density matrix $\rho_A$, and hence completely characterize the entanglement in an overall pure, bipartite state [21,22]. Therefore, the entanglement timescale (3) is a universal feature of bipartite entanglement.

The most common measure of entanglement, the entanglement entropy $S(\rho_A) = -\text{tr}_A(\rho_A \ln \rho_A)$, corresponds to the $\alpha \to 1$ limit of (4). Its second time derivative can be obtained by an analytic continuation in $\alpha$ from our general results for $\alpha \geqslant 2$ after which (3) appears with a logarithmically divergent

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prefactor, reflecting the sensitivity of $S(\rho_A)$ to small eigenvalues of the density matrix. We provide an example of these results by working with the Jaynes-Cummings model [23].

Notably, the entanglement timescale can be computed without the need to solve for the dynamics of the system. For a given experimental preparation of an unentangled state, our results provide an easily calculable estimate of when entanglement will become significant. Advances in the optical control of atoms have led to the first direct measurement of a Rényi entropy in a many-body system, and subsequently to measurements of its growth [24–28]. We return to these measurements for comparison to the entanglement timescale in Sec. IV.

## II. THE ENTANGLEMENT TIMESCALE FOR RÉNYI ENTROPIES

To begin, we briefly review the relevant properties of $S_\alpha(\rho_A)$ defined in (4). For any positive integer $\alpha$, $S_\alpha(\rho_A)$ is an entanglement measure that is minimized at zero if and only if the total state $\rho = |\Psi(t)\rangle\langle\Psi(t)|$ is separable. When $\rho$ represents a pure bipartite system, the Rényi entropies of its subsystems are equal, $S_\alpha(\rho_A) = S_\alpha(\rho_B)$. The Rényi entropies form a monotonically decreasing series in $\alpha$ since $\partial S_\alpha/\partial\alpha \leqslant 0$.

In the remainder of this section, we derive an entanglement timescale for the Rényi entropies of a pure bipartite state (2) evolving under a general Hamiltonian (1). Initially the subsystems are pure, $\rho_A = \rho_A^2$, because (2) is separable and therefore $S_\alpha(\rho_A)|_{t=0} = \frac{1}{1-\alpha}\ln \mathrm{tr}_A \rho_A^\alpha|_{t=0} = 0$. As the state evolves, the interactions between subsystems will generate entanglement. Starting at a minimum of $S_\alpha$, the first time derivative is initially zero. We will calculate the second derivative to obtain a Taylor expansion around $t = 0$ of the form

$$S_\alpha(\rho_A) = C_\alpha \frac{t^2}{T_{\mathrm{ent}}^2} + O(t^3). \tag{5}$$

We will find that the entanglement timescale $T_{\mathrm{ent}}$ takes the same form for all Rényi entropies, with $C_\alpha$ a constant.

Since the Rényi entropies are initially minimal, their first derivatives must vanish. We find $\frac{d}{dt}S_\alpha(\rho_A) = \frac{\alpha}{1-\alpha}(\mathrm{tr}_A\rho_A^\alpha)^{-1}\mathrm{tr}_A[(\mathrm{tr}_B\rho)^{\alpha-1}\mathrm{tr}_B(\frac{\partial\rho}{\partial t})]$. Note that in general, $[\mathrm{tr}_B(\partial\rho/\partial t), \mathrm{tr}_B\rho] \neq 0$. However, inside the $A$ trace, we can cyclically permute each term produced by the derivative into a common ordering as shown. Using the von Neumann equation $\partial\rho/\partial t = -i[H,\rho]$ with $\hbar = 1$ and using (1) and (2) in the $t = 0$ limit, we find $\frac{d}{dt}S_\alpha(\rho_A)|_{t=0} = \frac{i\alpha}{\alpha-1}(\mathrm{tr}_A\rho_A^\alpha)^{-1}\sum_n \mathrm{tr}_B(\rho_B B_n)\mathrm{tr}_A(\rho_A^{\alpha-1}A_n\rho_A - \rho_A^\alpha A_n) = 0$.

The leading order of the time evolution comes from the second derivative,

$$\frac{d^2}{dt^2}S_\alpha(\rho_A) = \frac{1}{1-\alpha}\left(\left(\mathrm{tr}_A\rho_A^\alpha\right)^{-1}\mathrm{tr}_A\left\{\frac{d^2}{dt^2}[\mathrm{tr}_B\rho(t)]^\alpha\right\} - \left(\mathrm{tr}_A\rho_A^\alpha\right)^{-2}\left\{\mathrm{tr}_A\frac{d}{dt}[\mathrm{tr}_B\rho(t)]^\alpha\right\}^2\right). \tag{6}$$

The second term vanishes when the $t \to 0$ limit is taken; this was the result of the first derivative calculation. We are left with the first term of (6) for which we find

$$\mathrm{tr}_A\left\{\frac{d^2}{dt^2}[\mathrm{tr}_B\rho(t)]^\alpha\right\} = \alpha\mathrm{tr}_A\left[(\mathrm{tr}_B\rho)^{\alpha-1}\mathrm{tr}_B\frac{\partial^2\rho}{\partial t^2} + \sum_{\beta=0}^{\alpha-2}(\mathrm{tr}_B\rho)^\beta\mathrm{tr}_B\frac{\partial\rho}{\partial t}(\mathrm{tr}_B\rho)^{\alpha-2-\beta}\mathrm{tr}_B\frac{\partial\rho}{\partial t}\right]. \tag{7}$$

The $\beta$ sum keeps track of the noncommuting factors which cannot be permuted into a common ordering. Applying the von Neumann equation leads to

$$\begin{aligned}\frac{d^2}{dt^2}S_\alpha(\rho_A)\big|_{t=0} = \frac{\alpha}{\alpha-1}\left(\mathrm{tr}_A\rho_A^\alpha\right)^{-1}\sum_{n,m}\Bigg[&\mathrm{tr}_B(B_nB_m\rho_B)\mathrm{tr}_A\left(2A_nA_m\rho_A^\alpha - 2A_m\rho_AA_n\rho_A^{\alpha-1}\right)\\ &+\mathrm{tr}_B(B_n\rho_B)\mathrm{tr}_B(B_m\rho_B)\sum_{\beta=0}^{\alpha-2}\mathrm{tr}_A\left(2\rho_A^{\beta+1}A_n\rho_A^{\alpha-\beta-1}A_m - \rho_A^\beta A_n\rho_A^{\alpha-\beta}A_m - \rho_A^{\beta+2}A_n\rho_A^{\alpha-2-\beta}A_m\right)\Bigg].\end{aligned} \tag{8}$$

Before simplifying (8) for general $\alpha$, it is useful to look at the unique case of $\alpha = 2$ which corresponds to the purity studied in [19]. In this case, the $\beta$ sum contains only a single term. Using the assumption of purity at $t = 0$ allows us to write

$$\frac{d^2}{dt^2}S_2(\rho_A)\big|_{t=0} = 4\sum_{n,m}[\mathrm{tr}_B(B_nB_m\rho_B) - \mathrm{tr}_B(B_n\rho_B)\mathrm{tr}_B(B_m\rho_B)][\mathrm{tr}_A(A_nA_m\rho_A) - \mathrm{tr}_A(A_m\rho_AA_n\rho_A)]. \tag{9}$$

Note that we have *not* assumed that $[A_n, A_m] = 0$. Instead, we have used the symmetry of $\mathrm{tr}_B(B_n\rho_B)\mathrm{tr}_B(B_m\rho_B)$ in the $n$, $m$ indices to exchange $A_n$ and $A_m$. Indeed, (9) exactly matches the main result of [19] when we account for the difference in the definitions of the purity and Rényi entropy. Defining the $\alpha$ purity, $P_\alpha(\rho_A) = \mathrm{tr}_A\rho_A^\alpha$, we have under our assumptions $\frac{d^2}{dt^2}S_\alpha(\rho_A)|_{t=0} = \frac{1}{1-\alpha}\frac{d^2}{dt^2}P_\alpha(\rho_A)|_{t=0}$.

Returning to the general case, it is possible to greatly simplify (8) by using the idempotency of $\rho_A(t=0)$, and $\rho_A^0 = \mathbb{I}_A$ where $\mathbb{I}_A$ is the identity operator for subsystem $A$. The special case of $\rho_A^0 = \mathbb{I}_A$ only occurs in the $\beta$ sum when $\beta$ takes on its extreme

values of 0 and $\alpha-2$. Each other term in the sum vanishes. The general result for $\alpha>2$ is

$$\frac{d^2}{dt^2}S_\alpha(\rho_A)\Big|_{t=0}=\frac{2\alpha}{\alpha-1}\sum_{n,m}[\mathrm{tr}_B(B_nB_m\rho_B)-\mathrm{tr}_B(B_n\rho_B)\mathrm{tr}_B(B_m\rho_B)][\mathrm{tr}_A(A_nA_m\rho_A)-\mathrm{tr}_A(A_m\rho_AA_n\rho_A)]$$
$$=\frac{2\alpha}{\alpha-1}\sum_{n,m}[\langle B_nB_m\rangle-\langle B_n\rangle\langle B_m\rangle][\langle A_nA_m\rangle-\langle A_n\rangle\langle A_m\rangle]=\frac{2\alpha}{\alpha-1}T_{\mathrm{ent}}^{-2}, \tag{10}$$

where we have used the simplification $\mathrm{tr}_A(A_m\rho_AA_n\rho_A)=\mathrm{tr}_A(A_m\rho_A)\mathrm{tr}_A(A_n\rho_A)$ for pure $\rho_A$ as shown in [19].

Equation (10) is our main result and shows that the second derivative of every Rényi entropy for $\alpha>2$ is of the same universal form as the $\alpha=2$ case studied previously. In fact, the coefficient incorporates the $\alpha=2$ case in Eq. (9) as well. The only remaining case is $\alpha=1$, which we turn to now.

The entanglement entropy $S(\rho_A)=-\mathrm{tr}_A(\rho_A\ln\rho_A)$ is the most widely used entanglement measure in the literature. It corresponds to the $\alpha\to1^+$ limit of $S_\alpha(\rho_A)$ after an analytic continuation in $\alpha$ [3,4]. Inserting $\alpha=1$ at intermediate steps in the derivation leading to (10) produces ill-defined quantities since the density matrix $\rho_A(t=0)$ is pure, and therefore singular. Nevertheless, we emphasize that inverse powers of $\rho_A$ do not appear in the final result (10). The prefactor $2\alpha/(\alpha-1)$ can be analytically continued in $\alpha$ and is analytic away from the simple pole at $\alpha=1$. Taking the limit of $2\alpha/(\alpha-1)$ as $\alpha\to1^+$ along the real axis shows that $d^2S(\rho_A)/dt^2|_{t=0}$ is proportional to the entanglement timescale with a divergent prefactor. This reflects the entanglement entropy's sensitivity to small eigenvalues of $\rho_A$ via the logarithm.

To make this point more clear, let $p_i(t)$ be the eigenvalues of $\rho_A$ such that $p_1(0)=1$ and $p_j(0)=0\,(j\neq1)$. Then the second derivative of the entanglement entropy, $S(\rho_A)=-\sum(p_i\ln p_i)$, in the $t\to0$ limit is

$$\frac{d^2S}{dt^2}=-\frac{d^2p_1}{dt^2}-\sum_{j\neq1}\left[(\ln p_j+3)\frac{d^2p_j}{dt^2}\right]. \tag{11}$$

Generically, $\lim_{t\to0}(d^2p_j/dt^2)\ln p_j$ is divergent since $d^2p_j/dt^2$ is not required to be zero initially. Still, the divergence of $d^2S/dt^2$ at $t=0$ does not imply that the entanglement entropy itself diverges; on the contrary, $S(\rho_A)$ is strictly bounded above by the dimension of the Hilbert space of subsystem $A$. Rather, $d^2S/dt^2$ appears in the Taylor series as the coefficient of $t^2$ which tames the logarithmic divergence. It should be noted that higher derivatives also diverge logarithmically at $t=0$, but are suppressed by higher powers of $t$.

## III. EXAMPLE: JAYNES-CUMMINGS MODEL

Equation (11) shows that the divergence of $d^2S/dt^2$ at $t=0$ for an initially pure product state found in (10) is not an artifact of the analytic continuation in $\alpha$. This is the generic behavior of the entanglement entropy for an initially separable state. To explore the physical significance of the entanglement timescale and to check the divergence of $d^2S/dt^2|_{t=0}$, we work with the Jaynes-Cummings model (JCM) of a two-level atom interacting with a quantized radiation field [23,29]. This system has been extensively studied in quantum optics because of its interesting entanglement properties [1,30] and quantum revivals [31,32]. In this section, we calculate the entanglement timescale for initially separable states, first by finding an analytic solution for the Rényi entropies at all times and then by studying the expectation values of the interaction terms in the initial state as dictated by (10). We explicitly show that the divergence of $d^2S/dt^2|_{t=0}$ is only logarithmic.

In the rotating-wave approximation, the JCM Hamiltonian is [23]

$$\frac{H}{\hbar}=\frac{\omega_0}{2}\sigma_z+\omega a^\dagger a+\lambda(a^\dagger\sigma_-+a\sigma_+). \tag{12}$$

Here, $\omega_0$ is the atomic transition frequency, $\omega$ is the characteristic field frequency, and $\lambda$ is a coupling constant. For simplicity, we impose the resonance condition $\omega=\omega_0$ and set $\hbar=1$. The Pauli operators can be written in terms of the atomic ground state $|g\rangle$ and excited state $|e\rangle$ as $\sigma_z=|e\rangle\langle e|-|g\rangle\langle g|$, $\sigma_-=|g\rangle\langle e|$, and $\sigma_+=|e\rangle\langle g|$. The field mode has a Fock basis $|n\rangle$ on which the creation and annihilation operators $a^\dagger$, $a$ act in the usual way. Notice that this Hamiltonian is of the assumed product form (1) and is time independent.

Let the overall initial state be the product of an arbitrary atomic state $|\psi\rangle_A=C_g|g\rangle+C_e|e\rangle$ and field state $|\psi\rangle_F=\sum_{n=0}^\infty C_n|n\rangle$. Then the overall state at any time is [2]

$$|\Psi(t)\rangle$$
$$=\sum_{n=0}^\infty\{[C_eC_n\cos(\lambda\sqrt{n+1}t)-iC_gC_{n+1}\sin(\lambda\sqrt{n+1}t)]|e\rangle$$
$$+[-iC_eC_{n-1}\sin(\lambda\sqrt{n}t)+C_gC_n\cos(\lambda\sqrt{n}t)]|g\rangle\}|n\rangle\,, \tag{13}$$

which is entangled for most times. Since the exact solution for the state is available, the Rényi entropies can be calculated directly for either subsystem after a partial trace. When the atom is initially excited ($C_e=1$, $C_g=0$),

$$\frac{d^2}{dt^2}S_\alpha(\rho_A)\Big|_{t=0}=\frac{2\alpha}{\alpha-1}\lambda^2\left[\sum_{n=0}^\infty(n+1)|C_n|^2-\sum_{n,m=0}^\infty\sqrt{m+1}\sqrt{n+1}C_{n+1}^*C_nC_{m+1}C_m^*\right]. \tag{14}$$

For comparison, if the atom is initially in the ground state, then the result in (14) changes slightly by the replacement $|C_n|^2\to|C_{n+1}|^2$ in the first sum.

The entanglement timescale can alternatively be computed from the Hamiltonian and initial state by using the definition in (10). This is much simpler because it does not require solving for the time evolution of the system. When the atom is initially

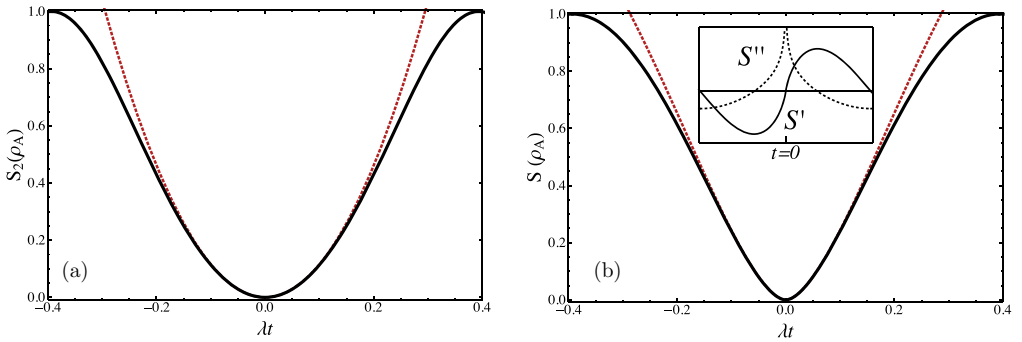


FIG. 1. (a) $S_2(\rho_A)$ for the Fock state with $N = 3$ and $C_e = 1$ is sinusoidal and $C^\infty$ smooth. $S_2$ is compared to the quadratic approximation with timescale $\lambda T_{\text{ent},e} = 1/4$ (dashed red line). (b) $S(\rho_A)$ for the same state is differentiable, but $d^2S/dt^2$ is discontinuous at $t = 0$ (inset, dashed line). Units of ln(2) are used in all figures.

excited, the only nonzero term in (10) is

$$T_{\text{ent},e}^{-2} = \lambda^2(\langle aa^\dagger\rangle - \langle a\rangle\langle a^\dagger\rangle)(\langle\sigma_+\sigma_-\rangle - \langle\sigma_+\rangle\langle\sigma_-\rangle)$$
$$= \lambda^2\left[\sum_{n=0}^{\infty}(n+1)|C_n|^2 - \sum_{n,m=0}^{\infty}\sqrt{m+1}\sqrt{n+1}C_{n+1}^* C_n C_{m+1} C_m^*\right] \geqslant 1. \tag{15}$$

Similarly for the ground-state case, we find a single nonzero term,

$$T_{\text{ent},g}^{-2} = \lambda^2(\langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle)(\langle\sigma_-\sigma_+\rangle - \langle\sigma_-\rangle\langle\sigma_+\rangle) \geqslant 0, \tag{16}$$

which is like (15) but with $|C_n|^2 \to |C_{n+1}|^2$ in the first sum.

The growth of entanglement is always controlled by the strength of the coupling $\lambda$ between subsystems. Indeed, it was pointed out in early studies of the JCM that $\lambda^{-1}$ is proportional to the time period over which the reduced states remain approximately pure [2]. The positivity of Rényi entropies requires that $T_{\text{ent}}^{-2}$ is positive. This is ensured by the results of [19], but can be seen here as a consequence of the Cauchy-Schwarz inequality which implies $\langle a^\dagger a\rangle \geqslant \langle a^\dagger\rangle\langle a\rangle$, etc.

From these general expressions, we can easily examine the growth of entanglement for some common field states. Consider when the field is initially in a Fock state, $|\psi\rangle_F = |N\rangle$. For the initially excited state, we find $T_{\text{ent},e} = (\lambda\sqrt{N+1})^{-1}$ and for the ground state, $T_{\text{ent},g} = (\lambda\sqrt{N})^{-1}$. Figure 1 shows $S_2(\rho_A)$ and $S(\rho_A)$ for $C_e = 1$, $C_g = 0$, and $N = 3$, along with the quadratic timescale approximation. Whereas $S_\alpha(\rho_A)$ for $\alpha \geqslant 2$ is $C^\infty$ smooth in this example, we see that $d^2S(\rho_A)/dt^2$ diverges at $t = 0$ as expected, while $dS(\rho_A)/dt$ is continuous at $t = 0$.

Instead, if the field starts in a coherent state,

$$|\psi\rangle_F = e^{-\frac{1}{2}|\nu|^2}\sum_{n=0}^{\infty}\frac{\nu^n}{\sqrt{n!}}|n\rangle, \quad a|\psi\rangle_F = \nu|\psi\rangle_F, \tag{17}$$

then the excited state timescale is $T_{\text{ent},e} = 1/\lambda$, whereas for the ground state, $T_{\text{ent},g}^{-1} = 0$. Notably, these timescales are independent of $\nu$. Figure 2 shows $S_2(\rho_A)$ and $S(\rho_A)$ for the coherent state with $\nu = 3$ and $C_e = 1$, $C_g = 0$. Once again, $d^2S(\rho_A)/dt^2$ diverges at $t = 0$, while $dS(\rho_A)/dt$ is continuous at $t = 0$.

For comparison, the coherent state with $\nu = 3$ and $C_e = 0$, $C_g = 1$ remains effectively separable for some time, as shown in Fig. 3. The divergence of the entanglement timescale in this case means one must look to higher orders in the Taylor expansion of $S_\alpha(t)$ to see the growth of entanglement. This is one example of an initial state where the correlated quantum uncertainty defined in [19] vanishes.

Equation (14) shows that the second time derivative of the entanglement entropy typically will be divergent in separable states. This is not a flaw of taking the $\alpha \to 1$ limit of the Rényi entropy, but is the actual behavior of the entanglement

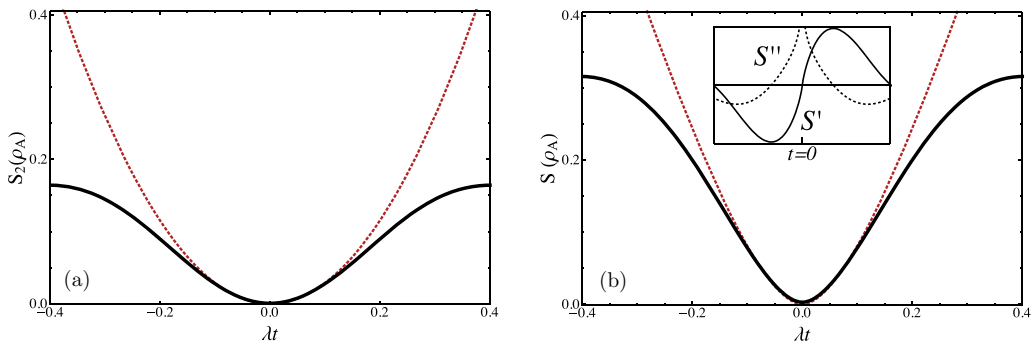


FIG. 2. (a) $S_2(\rho_A)$ for the coherent state with $\nu = 3$ and $C_e = 1$. The small-$t$ behavior is independent of $\nu$ and described by the quadratic timescale $\lambda T_{\text{ent},e} = 1$ (dashed red line). (b) $S(\rho_A)$ for the same state is differentiable, but $d^2S/dt^2$ is discontinuous at $t = 0$ (inset, dashed line).

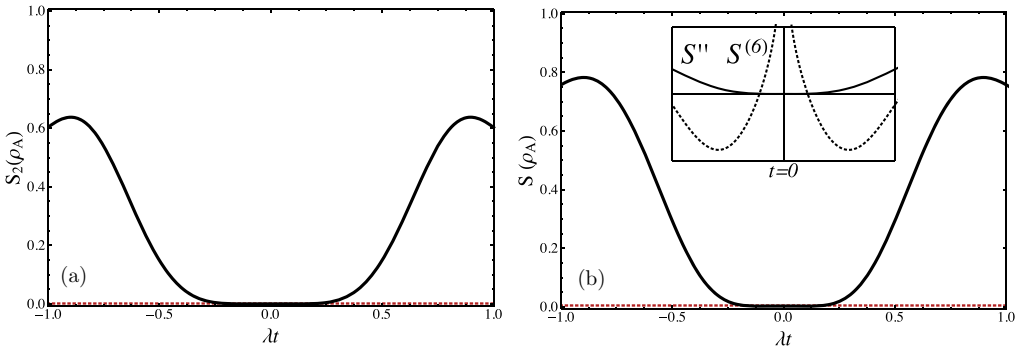


FIG. 3. (a) $S_2(\rho_A)$ for the coherent state with $\nu = 3$ and $C_g = 1$, where $T_{\mathrm{ent},g}^{-1} = 0$ indicates that the state remains effectively separable for a significant time. The leading behavior around $t = 0$ is sixth order in $t$. (b) $S(\rho_A)$ for the same state is $C^5$ smooth, with $d^2S/dt^2|_{t=0} = 0$ (inset, solid line) and $d^6S/dt^6$ discontinuous at $t = 0$ (inset, dashed line).

entropy. From the state (13), we can calculate the entanglement entropy directly for all times by diagonalizing the reduced density matrix of the atom $\rho_A(t)$ and finding its eigenvalues, $p_1(t) = \frac{1}{2}[1 + |\vec{s}(t)|]$, and $p_2(t) = \frac{1}{2}[1 - |\vec{s}(t)|]$ in terms of the Bloch vector $\vec{s}(t)$ [33]. For instance, starting with the atom in its excited state, we find $d^2p_1/dt^2|_{t=0} = -2T_{\mathrm{ent},e}^{-2} = -d^2p_2/dt^2|_{t=0}$. Using (11) leads to the logarithmically divergent result,

$$\left.\frac{d^2S}{dt^2}\right|_{t=0} = 2\left[-2+\ln 2-\lim_{t\to 0}\ln\left(1-\sum_{n=0}^{\infty}|C_n|^2\cos^2(\lambda\sqrt{n+1}t)\right)\right]T_{\mathrm{ent},e}^{-2}. \tag{18}$$

A similar logarithmic divergence occurs for the atom initially in its ground state.

## IV. DISCUSSION

The main result of [19] showed that for any unentangled pure bipartite state evolving under an arbitrary Hamiltonian, the growth of entanglement is characterized by a timescale which takes the universal form

$$T_{\mathrm{ent}} = \left[\sum_{n,m}(\langle A_nA_m\rangle - \langle A_n\rangle\langle A_m\rangle)(\langle B_nB_m\rangle - \langle B_n\rangle\langle B_m\rangle)\right]^{-\frac{1}{2}} \tag{19}$$

where entanglement is measured by the purity of subsystems. In this paper, we have shown that the same timescale characterizes the growth of entanglement as measured by any Rényi entropy. Since the family of Rényi entropies constitutes a complete determination of the entanglement in a pure bipartite system, the entanglement timescale universally describes the initial growth of bipartite entanglement.

It is easy to prove that the entanglement timescale obeys several properties expected of the Rényi entropy. As shown in [19], $T_{\mathrm{ent}}^2$ is a manifestly positive quantity so that the Rényi entropies initially increase from their minimum value. It is also symmetric between the subsystems $A$ and $B$ which reflects the symmetry $S_\alpha(\rho_A) = S_\alpha(\rho_B)$ for overall pure states. Furthermore, the coefficient $2\alpha/(\alpha - 1)$ in (10) is monotonically decreasing in $\alpha$, which is required by the general condition $\partial S_\alpha/\partial\alpha \leqslant 0$.

Rényi entropies are widely used theoretically and have recently been measured in isolated many-body systems [26], including their time dependence after an interaction is turned on [27]. The first such measurement was performed on a Bose-Einstein condensate trapped in an optical lattice and evolving under the Bose-Hubbard Hamiltonian in one dimension,

$$H = -J\sum_{\langle i,j\rangle}a_i^\dagger a_j + \frac{U}{2}\sum_i a_i^\dagger a_i(a_i^\dagger a_i - 1). \tag{20}$$

The first sum is over nearest-neighbor pairs and represents tunneling between neighboring sites at a rate $J$. The second sum over each lattice site represents the attractive energy among bosons sharing a site. In the experiment [27], a product of one-particle Fock states was prepared on six adjacent lattice sites with a barrier on each end. After a quench in which the interaction in (20) was turned on, the second Rényi $S_2(\rho_A)$ was measured in time for all unique partitions of the six sites.

The only interaction term in (20) that couples $A$ to $B$ is $-J(a_i^\dagger a_{i+1} + a_i a_{i+1}^\dagger)$, where sites $i$ and $i+1$ are neighbors across the partition. Thus, for any nontrivial partitioning, the entanglement timescale is the same, $T_{\mathrm{ent},BH}^{-2} = J^2\langle 1|a_i^\dagger a_i|1\rangle\langle 1|a_{i+1}a_{i+1}^\dagger|1\rangle + J^2\langle 1|a_i a_i^\dagger|1\rangle\langle 1|a_{i+1}^\dagger a_{i+1}|1\rangle = 4J^2$. Using the experimental value of $J/2\pi = 66$ Hz, we can estimate that the entanglement will become significant within a time $T_{\mathrm{ent},BH} = 1.2$ ms, which agrees with the experimental result displayed in Fig. 3 of Ref. [27]. This comparison is only approximate since the actual initial states prepared in the experiment were not free of entanglement.

The original motivation to determine the entanglement timescale was to estimate how quickly a generic quantum system will decohere due to entanglement with gravitational degrees of freedom [17,34,35]. This question is relevant to the black-hole information problem [36,37], where the Hawking quanta escaping from the black-hole horizon region may entangle with the geometry itself. To make any concrete statements about entanglement with gravitational degrees of freedom, one needs to work with quantum field theory or, better yet, quantum gravity. Since our derivation of the entanglement timescale assumes that the initial state is pure and unentangled, it is difficult to generalize these results to quantum field theory,

where typical states are highly entangled on all scales [38–40]. UV divergent entanglements can be avoided by considering the entanglement difference between states, for example with the relative entropy, which lends hope for our analysis of $d^2S_\alpha/dt^2$ [41,42]. One can otherwise avoid divergences by considering causally separated subregions, but this comes at the cost of losing purity for the combined system [43]. Moreover, for gauge field theories, the Hilbert space does not factorize across spatial boundaries, invalidating our assumptions [44,45]. Still, the growth of entanglement in quantum field theory states is a major area of research in many-body, condensed-matter, and high-energy physics [46–49], and it would be interesting to develop an entanglement timescale in these regimes.

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